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# A note on the interaction between solitary waves in a singularly-perturbed Korteweg–de Vries equation

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**Abstract.** For the fifth-order Korteweg–de Vries equation with a small parameter multiplying the highest-derivative term it is known that solitary waves are non-local, and are accompanied by co-propagating oscillatory waves of small amplitude and short wavelength. Here we report on some preliminary results for mutual interactions of these waves. First we impose a quantization condition upon a chain of solitary waves to obtain a periodic solution. Then we compute the interaction force between two neighbouring waves and hence estimate the conditions for a bound state to occur.

## 1. Introduction

The fifth-order Korteweg–de Vries equation

$$u_t + 6uu_x + u_{xxx} + \epsilon^2 u_{xxxxx} = 0 \tag{1}$$

has recently attracted much attention as a model equation for the study of non-local solitary waves (see for instance Pomeau *et al* (1988), Boyd (1991) and Grimshaw and Joshi (1992)). Interest centres on the case when the parameter $\epsilon$ is small, when equation (1) can be regarded as providing a singular perturbation of the Korteweg–de Vries equation. It is now known that symmetric solitary-wave solutions of equation (1) are not localized, and are accompanied by co-propagating oscillatory waves of small amplitude $\alpha$ and large wavenumber $k$. Indeed, oscillatory waves with phase speed $c$ have the linear dispersion relation

$$c = -k^2 + \epsilon^2 k^4. \tag{2}$$

These can then co-propagate with solitary waves of positive phase speed $c$ whenever $k^2 > \epsilon^{-2}$.

In the afore-mentioned references and elsewhere it has been established that this resonance between a solitary wave and small-amplitude short waves in fact occurs and is described asymptotically by

$$u \sim u_s(x - ct) + \alpha \sin(k|x - ct| - \delta). \tag{3}$$

Here the solitary-wave part $u_s$ is given by

$$u_s \sim a \operatorname{sech}^2 \gamma(x - ct) + O(\epsilon^2) \tag{4}$$

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where

$$c = 4\gamma^2 + 16\epsilon^2\gamma^2 \qquad a = 2\gamma^2. \tag{5}$$

The leading term in $u_s$ is just the well known solitary wave solution of the Korteweg–de Vries equation. The amplitude $\alpha$ of the oscillatory waves is given by

$$\alpha \sim (b/\epsilon^2)\exp(-\pi/2\epsilon\gamma) \tag{6}$$

and we note that it is exponentially small as $\epsilon \to 0$. The phase $\delta$ of the oscillatory waves is related to the parameter $b$ in (6) by the formula

$$b\cos\delta = -\pi K \tag{7}$$

where $K$ is a known numerical constant ($K \approx -19.97$). It is convenient to let $\gamma$ be the parameter describing the solitary wave component $u_s$. In terms of $\gamma$, we can use (5) in (2) to obtain

$$\epsilon^2 k^2 = 1 + 4\epsilon^2\gamma^2 \tag{8}$$

and, in particular, we note that $k \sim \epsilon^{-1}$ as $\epsilon \to 0$. With $\gamma$ fixed the asymptotic expression (3) describes a one-parameter family of symmetric non-local solitary waves characterized by the phase shift $\delta$. Boyd (1991) has called these 'nanopterons' and drawn attention to the prevalence of non-local solitary waves of this kind in a variety of physical systems. Note that it is sufficient to consider $\delta$ only in the range $|\delta| < \pi/2$.

Our concern in this short article is to report some preliminary results on the interaction between such non-local solitary waves through their common 'pedestal' of oscillatory waves. First, in the remaining part of this section to follow, we show how a quantization of a chain of solitary waves can be imposed to produce a periodic solution of equation (1). Then, in the next section, we describe the interaction force between two neighbouring solitary waves.

Suppose we seek a periodic solution of equation (1) with period $L$. If $\gamma L \gg 1$, we seek an approximate solution of equation (1) in the form of an infinite chain of non-local solitary waves

$$u \sim \sum_{-\infty}^{\infty} u_s(x - ct - rL) + u_w \tag{9}$$

where $u_w$ denotes the small-amplitude oscillatory waves between each solitary wave peak. Thus, in $0 < x' < L$ for instance, for $\gamma x' \gg 1$ and $\gamma(L - x') \gg 1$, where $x' = x - ct$, we obtain from (3) that

$$u_w = a\sin(k[x - ct] - \delta) = -a\sin(k[x - ct - L] + \delta). \tag{10}$$

Here the first expression is derived from the solitary wave at $x' = 0$ and the second from that at $x' = L$. For these to be consistent, we obtain the quantization condition,

$$kL = (2s + 1)\pi + 2\delta \tag{11}$$

where $s$ is an arbitrary integer. The same result is obtained if we consider the interval $rL < x < (r + 1)L$. Since $\gamma L \gg 1$ and $k \sim \epsilon^{-1}$ it follows that $L \approx 2\pi s\epsilon$ and $2\pi\epsilon\gamma \gg 1$. (11) agrees with that derived by Boyd (1991) using similar arguments, and we note that Boyd has called the periodic solutions (9) 'nanopteroidal' waves. Like the non-local solitary waves, with the parameter $\gamma$ fixed they form a one-parameter family of waves characterized by the phase shift $\delta$.

## 2. Solitary-wave interactions

To determine the interaction force between these non-local solitary waves (3) we use the perturbation procedure of Karpman and Solov'ev (1981), which is equivalent in the present context to the more general formulation of Gorshkov and Ostrovsky (1981). Thus we consider two non-local solitary waves $u^{(1)}$ and $u^{(2)}$, each given asymptotically by (3) and located at $\xi_1(t)$ and $\xi_2(t)$ respectively. Then we put

$$u \sim u^{(1)}(x - \xi_1(t)) + u^{(2)}(x - \xi_2(t)) + \delta u \tag{12}$$

where $\delta u$ describes the small modification due to the interaction. Our strategy is to regard each solitary wave as characterized by a core with parameters $\gamma_1(t)$ and $\gamma_2(t)$ respectively, together with their co-propagating oscillatory tails with amplitudes $\alpha_1(t)$ and $\alpha_2(t)$ respectively. But it is necessary to suppose that the phase shifts $\delta$ are the same. (The more general case of different phase shifts would seem to require a lengthier analysis than that developed here.) To leading order each core is described by the Korteweg–de Vries solitary wave (see (4)), and we compute the interaction force on this core due to the core of the other solitary wave and its oscillatory wave field. Thus, to leading order, the phase speed of each solitary wave is (see (5))

$$\frac{\mathrm{d}\xi_i}{\mathrm{d}t} = 4\gamma_i^2 \tag{13}$$

while $\mathrm{d}\gamma_i/\mathrm{d}t$ is found from the perturbation procedure of Karpman and Solov'ev (1981). We shall not give details since the outcome is that

$$\gamma \frac{\mathrm{d}^2\xi_1}{\mathrm{d}t^2} = -\frac{\partial U}{\partial L} \tag{14}$$

with an analogous expression for $\mathrm{d}^2\xi_2/\mathrm{d}t^2$. Here $L = \xi_2 - \xi_1$ is the distance between the solitary waves, while $U(L)$ is the interaction potential given by

$$U(L) = -\int_{-\infty}^{\infty} 3u^{(1)2}u^{(2)}\,\mathrm{d}x. \tag{15}$$

The interaction potential is obtained from the nonlinear term in the Hamiltonian of the system (see Gorshkov and Ostrovsky 1981). The validity of this perturbation procedure requires that the solitary waves have nearly equal amplitudes (i.e. $\gamma_1 \approx \gamma_2$) and be widely separated ($\gamma_i L \gg 1$).

First we compute the contribution to the potential $U(L)$ from the solitary-wave core. This is given by

$$U_s(L) = -\int_{-\infty}^{\infty} 3u_s^2(x)u_s(x - L)\,\mathrm{d}x. \tag{16}$$

Using the leading-order term in $u_s$ (see(4)) we find that

$$U_s(L) \sim -256\gamma^5 \exp(-2\gamma L) \tag{17}$$

where here we suppose that $\gamma_1 \approx \gamma_2 \approx \gamma$. Hence the interaction force is

$$-\partial U_s/\partial L \sim -512\gamma^6 \exp(-2\gamma L) \tag{18}$$

which agrees with the result of Karpman and Solov'ev (1981). As is well known, this is a repulsive force.

Next we consider the interaction between the solitary-wave core and the 'pedestal' of oscillatory waves associated with the other solitary wave. This gives a contribution to the potential of

$$U_{\mathrm{w}}(L) = -\int_{-\infty}^{\infty} 3u_s^2(x)u_{\mathrm{w}}(x-L)\,\mathrm{d}x. \tag{19}$$

Here we again approximate $u_s$ by its leading term, while $u_{\mathrm{w}}$ is found from (3). Since only the region near the solitary-wave core need be considered in (19) we find that

$$u_{\mathrm{w}} \sim -\alpha \sin\{k(x-L)+\delta\} \tag{20}$$

where we suppose that $\alpha_1 \approx \alpha_2 \approx \alpha$ and $\alpha$ is given by (6). Hence we find that, after some simplification,

$$U_{\mathrm{w}}(L) \sim -\frac{2\gamma^2\alpha}{\epsilon^2}\sin(kL-\delta)\int_{-\infty}^{\infty}\mathrm{sech}^2\,\gamma x\cos kx\,\mathrm{d}x \tag{21}$$

or

$$U_{\mathrm{w}}(L) \sim -\frac{4\pi b}{\epsilon^5}\exp(-\pi/\epsilon\gamma)\sin(kL-\delta). \tag{22}$$

Hence the interaction force is

$$-\frac{\partial U_{\mathrm{w}}(L)}{\partial L} \sim -\frac{4\pi b}{\epsilon^6}\exp(-\pi/\epsilon\gamma)\cos(kL-\delta). \tag{23}$$

This is an oscillatory function of $L$ and achieves a maximum attractive value when $kL-\delta = (2s+1)\pi$ where $s$ is an arbitrary integer. Note that this differs by a term $\delta$ from the quantization condition (11), since we have neglected here the phase matching between the two 'pedestals'. But it is interesting to observe that if the quantization value of $L$ (11) is substituted into (23) then the interaction force (23) becomes

$$(4\pi^2|K|/\epsilon^6)\exp(-\pi/\epsilon\gamma) \tag{24}$$

which is attractive, and independent of the phase shift $\delta$. Comparing this with the repulsive force (18) we conclude that the attractive force is stronger if

$$\exp(-\pi/\epsilon\gamma) > 0.65(\epsilon\gamma)^6\exp(-2\gamma L) \tag{25}$$

where we have set $|K| = 19.97$. Considering only the dominant exponential terms we conclude that the attractive force is stronger if $\gamma L > \pi/2\epsilon\gamma$ or $L > 2\pi/\epsilon c$ where $c$ is the solitary-wave phase speed (see (5)). We deduce that bound states are possible provided that $L$ satisfies this inequality.

This mechanism for the formation of bound states differs from those described recently for dissipative wave perturbations (see, for instance, Malomed (1991)), in that here the oscillatory wave field does not decay with distance from the core. Indeed, it is not obvious that the perturbation procedure of Karpman and Solov'ev (1981) can be used in the present case since, strictly speaking, it requires the perturbations to the solitary wave to decay at infinity. A similar comment applies to the more general theory of Gorshkov and Ostrovsky (1981). A related difficulty is that in calculating $U_w(L)$ we have only retained the leading-order term for $u_s$, whereas a simple calculation shows that the higher-order terms will contribute to $U_w(L)$ at the same order of magnitude. For instance the integral of $\epsilon^2 \operatorname{sech}^4 \gamma x \cos kx$ will give a term comparable to the integral of $\operatorname{sech}^2 \gamma x \cos kx$, and so on. Our procedure is consistent with the perturbation procedure formalism as given by Karpman and Solov'ev (1981), but the difficulty just mentioned indicates that expression (22) for $U_w(L)$ may not be quantitatively valid. Nevertheless we believe it gives an indication of the nature of this term, and in particular it seems clear that its proportionality to $\exp(-\pi/\epsilon\gamma)\sin(kL-\delta)$ will remain if higher-order terms are included.

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